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EFFECTIVE MODULI OF PIEZOELECTRIC MATERIAL WITH MICROCAVITIES

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Abstract—The effective electroelastic moduli of microcavity-weakened piezoelectric plates are investigated by the dilute, self-consistent, Mori–Tanaka and differential micromechanics theories. The results of perturbed heat intensity, strain and electric field (SEF) due to the presence of voids are obtained for two-dimensional (2-D) piezoelectric plates with microcavities of various shapes, and then the above four micromechanics models can be established with the results. These models can be applicable to a wide range of microcavities such as ellipse, circle, crack, triangle, square and pentagon. Some numerical results are presented to illustrate the applicability of these models.

1. INTRODUCTION

The problem of stiffness reduction of engineering materials due to the development or presence of many microdefects, such as cavities, inclusions or cracks, is of scientific significance and engineering importance and has been the subject of many investigations. For isotropic elastic materials, Thorpe and Sen (1985) obtained the results for randomly oriented elliptical holes, their analysis was done in the approximation of the self-consistent scheme, which seems to overestimate the effective compliance. Zhao and Weng (1990) considered "tubular" elliptical inclusions for two orientational distributions. In the study of 2-D isotropic medium containing circular holes, it is discovered that the elastic Young's modulus of a body containing holes is independent of the Poisson's ratio of the matrix and the 2-D effective Poisson's ratio flows to a fixed point as the percolation threshold is reached (Day *et al.*, 1992; Jun and Jasiuk, 1993). More recently, Christensen (1993) explored the extensions of the CLM theorem for 3-D material with holes. Jasiuk *et al.* (1994) performed numerical simulations on a network of springs containing a polygonal hole and considered a dilute concentration of elliptical cavities and of the randomly oriented polygonal holes. The work of Kachanov *et al.* (1994) should also be mentioned. They developed a unified description covering both cavities and cracks. As to the piezoelectric materials, however, a relatively small number of work has been done for the effective moduli of general anisotropic materials with cavities.

In this paper we study the effective electroelastic moduli of 2-D material containing a set of holes with the same size and same orientation. This assumption is only for simplifying the ensuing calculation and easy to extend to the case of randomly oriented holes. First we derive the perturbed heat intensity, strain and electric field due to the presence of the voids. The derivation is based on the solution of elastic displacement and electric potential for an anisotropic plate with a hole (Qin *et al.*, 1996), and then several micromechanics models (dilute, self-consistent, Mori–Tanaka and differential methods) are presented by way of the above perturbed results. Some numerical results are obtained for the effective electroelastic moduli of a voided medium under plane strain conditions.

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2. BASIC EQUATIONS OF PLANE THERMOPIEZOELECTRICITY

In this section we recall briefly the governing equations of 2-D piezoelectricity. Let $\mathbf{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$, $\boldsymbol{\phi}$, $\mathbf{E}$, $\mathbf{D}$, θ, $\mathbf{q}$ and $\mathbf{H}$ be the elastic displacement, strain, stress, electric potential, electric field and electric displacement, temperature change, heat flux and heat intensity, respectively. Here and after, the bold type letter stands for vector or tensor. The constitutive relations for linear piezoelectric materials can be written in the form (Yu and Qin, 1996)

$$H_i = \rho_{ij} q_j \tag{1}$$

$$\varepsilon_{ij} = F_{ijkm}\sigma_{km} + g_{kij}D_k + \alpha_{ij}\theta \tag{2}$$

$$(-E_i) = g_{ijk}\sigma_{jk} - \beta_{ij}D_j + \lambda_i\theta \tag{3}$$

$$q_i = k_{ij}H_j \tag{4}$$

$$\sigma_{ij} = C_{ijmn}\varepsilon_{mn} + e_{nij}(-E_n) - \gamma_{ij}\theta \tag{5}$$

$$D_i = e_{imn}\varepsilon_{mn} - \kappa_{in}(-E_n) - \chi_i\theta \tag{6}$$

$$H_i = -\partial\theta/\partial x_i \tag{7}$$

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2 \tag{8}$$

$$(-E_i) = \phi_{,i} \tag{9}$$

where $\mathbf{C}$ and $\mathbf{F}$ are elastic stiffness and compliance, $\mathbf{g}$ and $\mathbf{e}$ the piezoelectric constants, $\boldsymbol{\beta}$ and $\boldsymbol{\kappa}$ the dielectric permittivity coefficients, $\boldsymbol{\alpha}$ and γ are thermal expansion and thermal-stress coefficient tensors, $\boldsymbol{\lambda}$ and $\boldsymbol{\chi}$ are pyroelectric constant vectors, $\mathbf{k}$ and $\boldsymbol{\rho}$ are constants of heat conduction and heat resistivity, respectively. In the constitutive equations $(-E_i)$ is used instead of E_i because it will allow the construction of a symmetric generalized linear response matrix which will prove to be advantageous.

Since (1)–(6) involved second-, third- and fourth-rank tensors, it is useful to represent them by the familiar two index-notation (Nye, 1957). For a plane strain model where the material is assumed to be transversely isotropic and x_3-axis chosen as the poling direction, the constitutive equations are now written as

$$\begin{Bmatrix} q_1 \\ q_3 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{13} \\ k_{31} & k_{33} \end{bmatrix} \begin{Bmatrix} H_1 \\ H_3 \end{Bmatrix} \tag{10}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_3 \\ \sigma_5 \\ D_1 \\ D_3 \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{13} & 0 & 0 & e_{31} \\ c_{13} & c_{33} & 0 & 0 & e_{33} \\ 0 & 0 & c_{44} & e_{15} & 0 \\ 0 & 0 & e_{15} & -\kappa_{11} & 0 \\ e_{31} & e_{33} & 0 & 0 & -\kappa_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_3 \\ 2\varepsilon_5 \\ -E_1 \\ -E_3 \end{Bmatrix} - \begin{Bmatrix} \gamma_{11} \\ \gamma_{33} \\ 0 \\ 0 \\ \chi_3 \end{Bmatrix} \theta \tag{11}$$

and

$$\begin{Bmatrix} H_1 \\ H_3 \end{Bmatrix} = \begin{bmatrix} \rho_{11} & \rho_{13} \\ \rho_{31} & \rho_{33} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_3 \end{Bmatrix} \tag{12}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_3 \\ 2\varepsilon_5 \\ -E_1 \\ -E_3 \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{13} & 0 & 0 & p_{31} \\ f_{13} & f_{33} & 0 & 0 & p_{33} \\ 0 & 0 & f_{44} & p_{15} & 0 \\ 0 & 0 & p_{15} & \beta_{11} & 0 \\ p_{31} & p_{33} & 0 & 0 & \beta_{33} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_3 \\ \sigma_5 \\ D_1 \\ D_3 \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{33} \\ 0 \\ 0 \\ \lambda_3 \end{Bmatrix} \theta \tag{13}$$

or in matrix form

$$\mathbf{q} = \mathbf{k}\mathbf{H}, \quad \mathbf{H} = \boldsymbol{\rho}\mathbf{q} \tag{14}$$

$$\Pi = \mathbf{E}\mathbf{Z} - \gamma\theta, \quad \mathbf{Z} = \mathbf{F}\Pi + \boldsymbol{\alpha}\theta \tag{15}$$

where

$$\Pi = \{\sigma_{11} \quad \sigma_{33} \quad \sigma_{13} \quad D_1 \quad D_3\}^T = \{\sigma_1 \quad \sigma_3 \quad \sigma_5 \quad D_1 \quad D_3\}^T,$$
$$\mathbf{Z} = \{Z_{11} \quad Z_{22} \quad 2Z_{12} \quad Z_{31} \quad Z_{32}\}^T = \{\varepsilon_1 \quad \varepsilon_3 \quad 2\varepsilon_5 \quad -E_1 \quad -E_3\}^T.$$

3. OVERALL CONSTITUTIVE RELATIONS

What follows is concerned with the piezoelectric analogue of the uncoupled theory of thermoelasticity where the electric and elastic fields are fully coupled, but the temperature enters the problem only through the constitutive equations. As a result of this, the effective conductivity and the effective electroelastic constants can be determined independently, while the evaluation on the effective thermal expansion and pyroelectric coefficients requires the information about both of them. The details are described as follows.

3.1. *Effective electroelastic moduli*

The effective electroelastic moduli of a voided body are defined as (Yu and Qin, 1996)

$$\bar{\Pi} = \mathbf{E}^*\bar{\mathbf{Z}} - \gamma^*\bar{\theta} \tag{16a}$$

or the equivalent

$$\bar{\mathbf{Z}} = \mathbf{F}^*\bar{\Pi} + \boldsymbol{\alpha}^*\bar{\theta} \tag{16b}$$

where the overbar denotes the area average of a quantity over a representative area element (RAE) Ω, i.e.,

$$\overline{(\cdot)} = \frac{1}{\Omega}\int_{\Omega} (\cdot)\, d\Omega \tag{17}$$

and the superscript "*" stands for the effective value.

3.1.1. *Perturbed SEF due to holes*. The micromechanics theories may be established based on some fundamental results in the theory of two-phase elastic media. In the case of two-phase materials, the area average of stresses, electric displacements, elastic displacements and electric potential are defined by

$$\bar{\Pi} = v_1\bar{\Pi}^{(1)} + v_2\bar{\Pi}^{(2)} \tag{18}$$

$$\bar{\mathbf{Z}} = v_1\bar{\mathbf{Z}}^{(1)} + v_2\bar{\mathbf{Z}}^{(2)} \tag{19}$$

where subscripts (or superscripts) "1" and "2" denote the matrix and inclusion phases, v_1 and v_2 their area fractions.

It is well-known, e.g., Dunn and Taya (1993), that when such a material is subjected to remote Π^0 or $\mathbf{Z}^0$ the effective moduli $\mathbf{E}^*$ and $\mathbf{F}^*$ can be expressed in the form

$$\mathbf{E}^* = \mathbf{E}_1 + (\mathbf{E}_2 - \mathbf{E}_1)\mathbf{A}_2 v_2 \tag{20}$$

$$\mathbf{F}^* = \mathbf{F}_1 + (\mathbf{F}_2 - \mathbf{F}_1)\mathbf{B}_2 v_2 \tag{21}$$

in which the symmetric tensors $\mathbf{A}_2$ and $\mathbf{B}_2$ are defined by the linear relations

$$\bar{\mathbf{Z}}^{(2)} = \mathbf{A}_2\mathbf{Z}^0, \quad \bar{\Pi}^{(2)} = \mathbf{B}_2\Pi^0 \tag{22}$$

Now we consider the case when the inclusion becomes void. This implies that $\mathbf{E}_2 \to 0$, $\mathbf{F}_2 \to \infty$. It should be pointed out that the assumption below has been adopted in our analysis. The voids under consideration are thought of as being filled with air, which has a dielectric constant approximately three orders of magnitude smaller than the dielectric constants of the piezoelectric material. The consequence of such an assumption is that the boundary conditions on the hole boundary are given by $\Pi \cdot \mathbf{m} = 0$, where $\mathbf{m}$ is outward normal to the hole boundary. This is also equivalent to setting $\mathbf{E}_2 = 0$, where $\mathbf{E}_2$ stands for the material constants of the hole-phase. The discussions on the validity of the electrical boundary conditions can be found in literature (Dunn, 1994; Parton and Kudryatvsev, 1988). Then (20) and (21) become

$$\mathbf{E}^* = \mathbf{E}_1(\mathbf{I} - \mathbf{A}_0 v_2) \tag{23}$$

$$\mathbf{F}^* = \mathbf{F}_1(\mathbf{I} + \mathbf{B}_0 v_2) \tag{24}$$

where $\mathbf{I}$ is the unit tensor, $\mathbf{A}_0$ is $\mathbf{A}_2$ of (20) for voids, and $\mathbf{B}_0$ is defined by

$$\bar{\mathbf{Z}}^{(2)} = \mathbf{F}_1\mathbf{B}_0\Pi^0 \tag{25}$$

The interpretation of $\bar{\mathbf{Z}}^{(2)}$ in (25) follows from the average strain theorem (Yu and Qin, 1996)

$$\bar{Z}_{ij}^{(2)} = \frac{1}{2\Omega_2}\int_{\partial\Omega_2} \{[1 + H(i-3)]U_i n_j + U_j n_i\}\,\mathrm{d}\Omega \tag{26}$$

where Ω_2 and $\partial\Omega_2$ are the total area and boundary of the voids, $\mathbf{n} = \{n_1 \quad n_2 \quad 0\}^T$ is the normal local to the void surface, $\mathbf{U} = \{U_1 \quad U_2 \quad U_3\}^T = \{u_1 \quad u_2 \quad \phi\}^T$, and $H(i)$ is the Heaviside step function.

The estimation of integral (26) and thus, $\mathbf{A}_0$ (or $\mathbf{B}_0$) is the key to predicting the effective electroelastic moduli $\mathbf{E}^*$ and $\mathbf{F}^*$. The approximation of integral (26) through use of various micromechanics models is the subject of the subsequent subsection.

To calculate the integral (26), consider a sheet containing a void. For a particular void, its contour is described by

$$x_1 = a(\cos\psi + \eta\cos\psi) \tag{27}$$

$$x_3 = a(c\sin\psi - \eta\sin k\psi) \tag{28}$$

where $0 < c \leqslant 1$, and k is an integer. By an appropriate selection of the parameters c, k and η, we can obtain various special kinds of voids, such as ellipse, square, and the like. When a set of far fields Π^0 is applied, the elastic displacement and electric potential at a point of the hole boundary has been obtained by Qin *et al.* (1998) as

$$\mathbf{U} = x_1\varepsilon_1^0 + x_3\varepsilon_3^0 + \{ac\mathbf{L}^{-1}\cos\psi - a\eta\mathbf{L}^{-1}\cos k\psi - ac\mathbf{SL}^{-1}\sin\psi + a\eta\mathbf{SL}^{-1}\sin k\psi\}\mathbf{t}_1^0 - \{a\mathbf{L}^{-1}\mathbf{S}^T\cos\psi + a\eta\mathbf{L}^{-1}\mathbf{S}^T\cos k\psi - a(\mathbf{H}+\mathbf{SL}^{-1}\mathbf{S}^T)(\sin\psi + \eta\sin k\psi)\}\mathbf{t}_3^0 \quad (29)$$

where

$$\mathbf{t}_1^0 = \{\sigma_{11}^0 \quad \sigma_{13}^0 \quad D_1^0\}^T, \qquad \mathbf{t}_3^0 = \{\sigma_{31}^0 \quad \sigma_{33}^0 \quad D_3^0\}^T,$$
$$\varepsilon_1^0 = \{\varepsilon_{11}^0 \quad \varepsilon_{13}^0 \quad -E_1^0\}^T, \qquad \varepsilon_3^0 = \{\varepsilon_{31}^0 \quad \varepsilon_{33}^0 \quad -E_3^0\}^T$$

$\varepsilon_{ij}^0 = F_{ijkm}\Pi_{km}^0$, and **S**, **L** and **H** are the well-known real matrices in the Stroh formalism, which is defined as (Qin *et al.*, 1998)

$$\mathbf{S} = i(2\mathbf{AB}^T - \mathbf{I}), \quad \mathbf{H} = 2i\mathbf{AA}^T, \quad \mathbf{L} = -2i\mathbf{BB}^T \quad (30)$$

The substituting (29) into (26) and integrating it along the whole contour of the void, one obtains

$$\bar{\mathbf{Z}}^{(2)} = \mathbf{Q}\Pi^0 \quad (31)$$

where **Q** is a 5×5 symmetric matrix whose components are

$$Q_{11} = f_{11}(c-k\eta^2) + (\mathbf{L}^{-1})_{11}(c^2+k\eta^2)$$
$$Q_{12} = (c-k\eta^2)[f_{13} - (\mathbf{L}^{-1}\mathbf{S}^T)_{12}]$$
$$Q_{13} = (c^2+k\eta^2)(\mathbf{L}^{-1})_{12} + (k\eta^2-c)(\mathbf{L}^{-1}\mathbf{S}^T)_{11}$$
$$Q_{14} = (c^2+k\eta^2)(\mathbf{L}^{-1})_{13}$$
$$Q_{15} = (c-k\eta^2)[p_{31} - (\mathbf{L}^{-1}\mathbf{S}^T)_{13}]$$
$$Q_{22} = (c-k\eta^2)f_{33} + (1+k\eta^2)[H_{22} + (\mathbf{SL}^{-1}\mathbf{S}^T)_{22}]$$
$$Q_{23} = (-c+k\eta^2)(\mathbf{SL}^{-1})_{22} + (k\eta^2+1)[H_{21} + (\mathbf{SL}^{-1}\mathbf{S}^T)_{21}]$$
$$Q_{24} = (-c+k\eta^2)(\mathbf{SL}^{-1})_{23}$$
$$Q_{25} = (c-k\eta^2)p_{33} + (k\eta^2+1)[H_{23} + (\mathbf{SL}^{-1}\mathbf{S}^T)_{23}]$$
$$Q_{33} = (c-k\eta^2)[f_{44} - 2(\mathbf{SL}^{-1})_{12} + (c^2+k\eta^2)(\mathbf{L}^{-1})_{22} + (1+k\eta^2)[H_{11} + (\mathbf{SL}^{-1}\mathbf{S}^T)_{11}]$$
$$Q_{34} = (k\eta^2-c)[(\mathbf{SL}^{-1})_{13} - p_{15}] + (k\eta^2+c^2)(\mathbf{L}^{-1})_{23}$$
$$Q_{35} = (k\eta^2+1)[(\mathbf{SL}^{-1}\mathbf{S}^T)_{13} + H_{13}] + (k\eta^2-c)(\mathbf{L}^{-1}\mathbf{S}^T)_{23}$$
$$Q_{44} = (k\eta^2+c^2)(\mathbf{L}^{-1})_{33} - (k\eta^2-c)\beta_{11}$$
$$Q_{45} = (k\eta^2-c)(\mathbf{L}^{-1}\mathbf{S}^T)_{33}$$
$$Q_{55} = (c-k\eta^2)\beta_{33} + (1+k\eta^2)[H_{33} + (\mathbf{SL}^{-1}\mathbf{S}^T)_{33}]$$

Thus, from (22), (25) and (31), we have

$$\mathbf{A}_0 = \mathbf{A}_0(\mathbf{E}_1, \mathbf{E}^*, \chi) = \mathbf{Q}\mathbf{E}_1 \tag{32}$$

$$\mathbf{B}_0 = \mathbf{B}_0(\mathbf{F}_1, \mathbf{F}^*, \chi) = \mathbf{F}_1^{-1}\mathbf{Q} \tag{33}$$

where χ symbolizes void geometry.

3.1.2. *Micromechanics models for effective electroelastic moduli.* In this subsection the results of (31)–(33) will be used to establish several micromechanics approximation theories for the effective electroelastic moduli.

Dilute scheme (DIL)

In the dilute approximation we assume that the interaction among the voids in an infinite plate can be ignored. The concentration factors $\mathbf{A}_0$ and $\mathbf{B}_0$ are then obtained from the solution of the auxiliary problem of a single void embedded in an infinite plate. Thus the concentration factors $\mathbf{A}_0^{\mathrm{DIL}}$ and $\mathbf{B}_0^{\mathrm{DIL}}$ are given by

$$\mathbf{A}_0^{\mathrm{DIL}} = \mathbf{A}_0(\mathbf{E}_1, \chi) = \mathbf{Q}(\mathbf{E}_1, \chi)\mathbf{E}_1 \tag{34}$$

$$\mathbf{B}_0^{\mathrm{DIL}} = \mathbf{B}_0(\mathbf{F}_1, \chi) = \mathbf{F}_1^{-1}\mathbf{Q}(\mathbf{E}_1, \chi) \tag{35}$$

The substitution of (34) and (35) into (23) and (24), we obtain

$$\mathbf{E}^{\mathrm{DIL}} = \mathbf{E}_1(\mathbf{I} - v_2\mathbf{Q}(\mathbf{E}_1, \chi)\mathbf{E}_1) \tag{36}$$

$$\mathbf{F}^{\mathrm{DIL}} = \mathbf{F}_1 + v_2\mathbf{Q}(\mathbf{E}_1, \chi) \tag{37}$$

Self-consistent methods (SC)

The essential assumption employed in the self-consistent method is that each void sees the effective medium of as yet unknown moduli. Thus, the concentration factors $\mathbf{A}_0^{\mathrm{SC}}$ and $\mathbf{B}_0^{\mathrm{SC}}$ are simply given by

$$\mathbf{A}_0^{\mathrm{SC}} = \mathbf{A}_0(\mathbf{E}^{\mathrm{SC}}, \chi) = \mathbf{Q}(\mathbf{E}^{\mathrm{SC}}, \chi)\mathbf{E}_1 \tag{38}$$

$$\mathbf{B}_0^{\mathrm{SC}} = \mathbf{B}_0(\mathbf{F}^{\mathrm{SC}}, \chi) = \mathbf{F}_1^{-1}\mathbf{Q}(\mathbf{E}^{\mathrm{SC}}, \chi) \tag{39}$$

when (34) combines with (23) it yields an implicit algebraic matrix equation for $\mathbf{E}^{\mathrm{SC}}$. In general an explicit analytical solution for $\mathbf{E}^{\mathrm{SC}}$ is impossible, and a numerical iteration is required. In the calculation, we take the dilute solution as the initial values of $\mathbf{E}^{\mathrm{SC}}$, and then enter the iterative scheme. The iteration will be terminated if the relative error of $\mathbf{E}^{\mathrm{SC}}$ between two adjacent iterations is less than a prescribed tolerance δ ($\delta = 0.001$ in our analysis).

Mori–Tanaka theory (MT)

The key assumption in the Mori–Tanaka theory (1973) is that $\mathbf{A}_0^{\mathrm{MT}}$ is given by the solution for a single void embedded in an intact plate subject to an applied strain field equal to the as yet unknown average field in the plate, which means that the introduction of voids in the plate results in a value of $\bar{\mathbf{Z}}^{(2)}$ given by

$$\bar{\mathbf{Z}}^{(2)} = \mathbf{Q}(\mathbf{E}_1, \chi)\mathbf{E}_1\bar{\mathbf{Z}}^{(1)} \tag{40}$$

With (18), (19), (22) and (40), the concentration factors $\mathbf{A}_0^{\mathrm{MT}}$ and $\mathbf{B}_0^{\mathrm{MT}}$ can be written in the form

$$\mathbf{A}_0^{MT} = \mathbf{Q}\mathbf{E}_1[v_1\mathbf{I}+v_2\mathbf{Q}\mathbf{E}_1]^{-1} \tag{41}$$

$$\mathbf{B}_0^{MT} = \mathbf{E}_1\mathbf{Q}[v_1\mathbf{I}+v_2\mathbf{E}_1\mathbf{Q}]^{-1} \tag{42}$$

The expressions (41) and (42) have the similar form as those of Dunn and Taya (1993) in treating two-phase matrix based piezoelectric composite. It can be seen from (41) and (42) that the Mori–Tanaka theory provides explicit expressions for effective electroelastic moduli of voided piezoelectric sheet.

Differential scheme (DS)

The essence of the differential scheme is the construction of the final voided medium from the intact material through successive replacement of an incremental area of the current voided material with that of the voids. Following Mclaughlin (1977) and Hashin (1988), the application of the differential scheme for voided plane piezoelectric medium can be obtained as

$$\frac{\mathrm{d}\mathbf{E}^{DS}}{\mathrm{d}v_2} = -\mathbf{E}^{DS}\mathbf{A}_0^{DS}/(1-v_2) \tag{43}$$

$$\mathbf{A}_0^{DS} = \mathbf{A}_0(\mathbf{E}^{DS}, \chi) \tag{44}$$

subjected to the initial conditions

$$\mathbf{E}^{DS}(v_2 = 0) = \mathbf{E}_1 \tag{45}$$

Equation (43) represents a set of 5×5 coupled nonlinear ordinary differential equations, which can be solved with some numerical methods, such as the well-known fourth order Runge–Kutta integration scheme.

3.2. *Effective conductivity*

3.2.1. *The concentration factors.* Similar to the above analysis for electroelastic fields, we have

$$\mathbf{k}^* = \mathbf{k}_1(\mathbf{I}-\mathbf{A}_0 v_2) \tag{46}$$

$$\boldsymbol{\rho}^* = \boldsymbol{\rho}_1(\mathbf{I}+\mathbf{B}_0 v_2) \tag{47}$$

where $\mathbf{B}_0$ is now defined by

$$\bar{\mathbf{H}}^{(2)} = \boldsymbol{\rho}_1\mathbf{B}_0\mathbf{q}^0 \tag{48}$$

The interpretation of $\bar{\mathbf{H}}^{(2)}$ in (48) in this case follows from the average intensity theorem (Hashin, 1983)

$$\bar{H}_i^{(2)} = \frac{1}{\Omega_2}\int_{\partial\Omega_2} \theta n_i\,\mathrm{d}c \tag{49}$$

the temperature change θ at a point of the void boundary has been given by Qin *et al.* (1998):

$$\theta = \frac{-a}{\tilde{k}}\{ch_1^0 \cos\psi + h_2^0 \sin\psi - \eta(h_1^0 \cos k\psi - h_2^0 \sin k\psi)\} \tag{50}$$

where $\tilde{k} = \sqrt{k_{11}k_{33} - k_{13}^2}$.

The substituting (50) into (49) and integrating it along the whole contour of the void, we have

$$\bar{\mathbf{H}}^{(2)} = \mathbf{Q}\mathbf{q}^0 \tag{51}$$

where $\mathbf{Q} = \mathbf{Q}(\mathbf{k}_1, \chi)$ is a 2×2 diagonal matrix whose components are

$$Q_{11} = \frac{c^2 + k\eta^2}{\tilde{k}(c - k\eta^2)}$$

$$Q_{22} = \frac{1 + k\eta^2}{\tilde{k}(c - k\eta^2)}$$

$$Q_{12} = Q_{21} = 0 \tag{52}$$

Thus, from (46)–(48) and (51), one sees

$$\mathbf{A}_0 = \mathbf{A}_0(\mathbf{k}_1, \mathbf{k}^*, \chi) = \mathbf{Q}\mathbf{k}_1 \tag{53}$$

$$\mathbf{B}_0 = \mathbf{B}_0(\mathbf{k}_1, \mathbf{k}^*, \chi) = \mathbf{k}_1\mathbf{Q} \tag{54}$$

where χ symbolizes void geometry again.

3.2.2. *Micromechanics models for effective conductivity.* In this subsection the results of (51)–(54) will be used to establish several micromechanics approximation theories for the effective heat conductivity. Similar to the previous analysis, the concentration tensors $\mathbf{A}_0$ and $\mathbf{B}_0$ for dilute and self-consistent methods are, respectively, given by

$$\mathbf{A}_0^{\mathrm{DIL}} = \mathbf{Q}(\mathbf{k}_1, \chi)\mathbf{k}_1, \quad \mathbf{B}_0^{\mathrm{DIL}} = \mathbf{k}_1\mathbf{Q}(\mathbf{k}_1, \chi) \tag{55}$$

$$\mathbf{A}_0^{\mathrm{SC}} = \mathbf{Q}(\mathbf{k}^{\mathrm{SC}}, \chi)\mathbf{k}_1, \quad \mathbf{B}_0^{\mathrm{SC}} = \mathbf{k}_1\mathbf{Q}(\mathbf{k}^{\mathrm{SC}}, \chi) \tag{56}$$

The substitution of (55) and (56) into (46) and (47), we have

$$\mathbf{k}^{\mathrm{DIL}} = \mathbf{k}_1(\mathbf{I} - v_2\mathbf{Q}\mathbf{k}_1), \quad \boldsymbol{\rho}^{\mathrm{DIL}} = \boldsymbol{\rho}_1(\mathbf{I} + v_2\mathbf{k}_1\mathbf{Q})$$

$$\mathbf{k}^{\mathrm{SC}} = \mathbf{k}_1(\mathbf{I} - v_2\mathbf{Q}^{\mathrm{SC}}\mathbf{k}_1), \quad \boldsymbol{\rho}^{\mathrm{SC}} = \boldsymbol{\rho}_1(\mathbf{I} + v_2\mathbf{k}_1\mathbf{Q}^{\mathrm{SC}}) \tag{57}$$

For Mori–Tanaka theory the perturbed heat intensity is now given by

$$\bar{\mathbf{H}}^{(2)} = \mathbf{Q}(\mathbf{k}_1, \chi)\mathbf{k}_1\bar{\mathbf{H}}^{(1)} \tag{58}$$

With (46)–(48) and (58), the concentration factors $\mathbf{A}_0^{\mathrm{MT}}$ and $\mathbf{B}_0^{\mathrm{MT}}$ can be written in the form

$$\mathbf{A}_0^{\mathrm{MT}} = \mathbf{Q}(\mathbf{k}_1, \chi)\mathbf{k}_1[v_1\mathbf{I} + v_2\mathbf{Q}(\mathbf{k}_1, \chi)\mathbf{k}_1]^{-1}$$

$$\mathbf{B}_0^{\mathrm{MT}} = \mathbf{k}_1\mathbf{Q}(\mathbf{k}_1, \chi)[v_1\mathbf{I} + v_2\mathbf{k}_1\mathbf{Q}(\mathbf{k}_1, \chi)]^{-1} \tag{59}$$

It can be seen from (59) that the Mori–Tanaka theory also provide explicit expressions for effective conductivity.

Finally, for the differential scheme, we have

$$\frac{\mathrm{d}\mathbf{k}^{\mathrm{DS}}}{\mathrm{d}v_2} = -\mathbf{k}^{\mathrm{DS}}\mathbf{A}_0^{\mathrm{DS}}/(1-v_2) \tag{60}$$

$$\mathbf{A}_0^{\mathrm{DS}} = \mathbf{A}_0(\mathbf{k}^{\mathrm{DS}}, \chi) = Q(\mathbf{k}^{\mathrm{DS}}, \chi)\mathbf{k}_1 \tag{61}$$

subjected to the initial conditions

$$\mathbf{k}^{\mathrm{DS}}(v_2 = 0) = \mathbf{k}_1 \tag{62}$$

Equation (60) represents a set of 2×2 coupled nonlinear ordinary differential equations, which can also be solved with some numerical methods, such as the well-known fourth order Runge–Kutta integration scheme.

4. NUMERICAL ANALYSIS

As illustrated we consider a voided $BaTiO_3$ (Dunn, 1993), the properties of which are given as follows

$$c_{11} = 150\,\mathrm{GPa},\quad c_{12} = 66\,\mathrm{GPa},\quad c_{13} = 66\,\mathrm{GPa},\quad c_{33} = 146\,\mathrm{GPa},\quad c_{44} = 44\,\mathrm{GPa},$$
$$e_{31} = -4.35\,\mathrm{C/m^2},\quad e_{33} = 17.5\,\mathrm{C/m^2},\quad e_{15} = 11.4\,\mathrm{C/m^2},\quad \kappa_{11} = 1115\kappa_0,$$
$$\kappa_{33} = 1260\kappa_0,\quad \kappa_0 = 8.85\times 10^{-12}\,\mathrm{C^2/Nm^2}$$

Figure 1 shows the plot of $c^*_{11}/c^{(1)}_{11}$ as a function of area fraction of holes v_2 for the voided piezoelectric ceramics, obtained by dilute, self-consistent, Mori–Tanaka, differential scheme and finite element (FE) method. In the finite element analysis, the configuration of

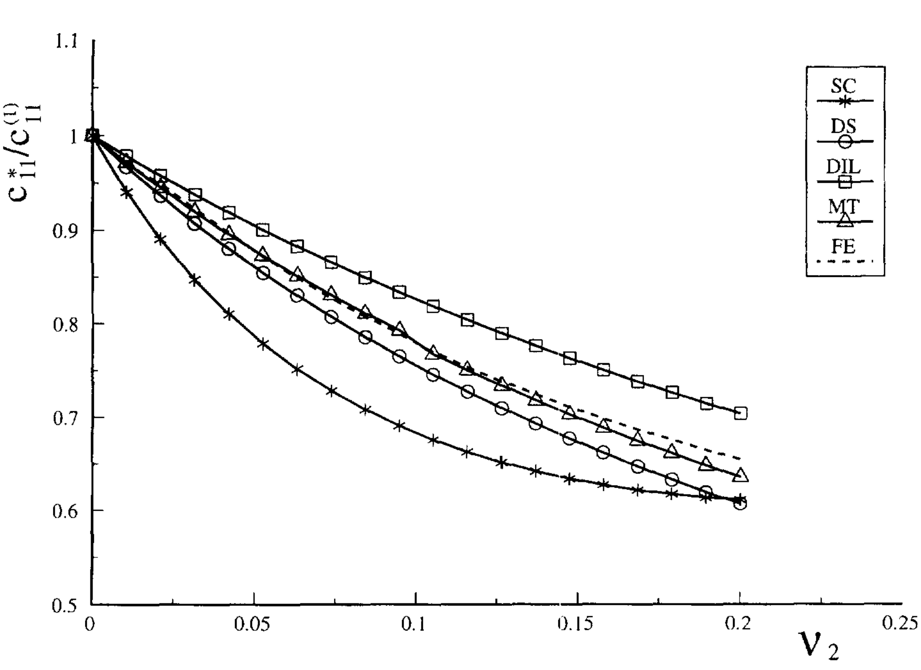

Fig. 1. Normalised modulus $c^*_{11}/c^{(1)}_{11}$ vs area fraction v_2.

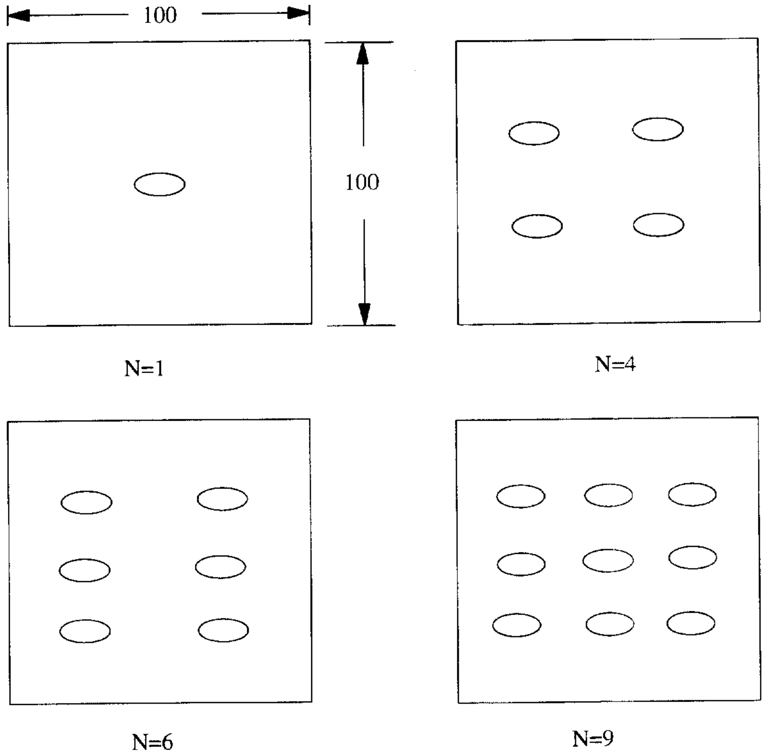

Fig. 2. Configuration for an RAE in FE analysis.

a RAE used is shown in Fig. 2 for $N = 1, 4, 6, 9$, where N is the hole number of a particular RAE. For simplicity, elliptical hole is chosen as a sample. The major and minor axes are 10 and 5, respectively. Thus the area fraction v_2 equals $5N\pi/1000$ (see Fig. 2). It is clearly observed that the dilute scheme overestimates the effective moduli than those from finite element analysis, while the self-consistent technique underestimates the corresponding values than those from the FE method. The results predicted by the Mori–Tanaka and the differential methods, on the other hand, are closest to the FE results. As far as we know, however, no reference results are available presently.

5. CONCLUSION

We have presented in this paper an application of micromechanics theories to the computation of effective electroelastic moduli of voided piezoelectric medium. These theories involve dilute, self-consistent, Mori–Tanaka and differential approximations. Based on the solutions of displacement and electric potential for an infinite medium with a hole, the perturbed heat conductivity and SEF due to the presence of the holes have been derived analytically and used to construct the concentration factor $\mathbf{A}_0$ for each micromechanics model. Finally, behavior of each of the micromechanics model used to calculate the effective electroelastic moduli has been examined. Although the results are confined to the case of plane strain and all holes with the same size and same orientation, it is easy to extend it to other plane problems, such as the case of $u_2 = u_2(x_1, x_3) \neq 0$ and the holes being randomly oriented. Moreover, this paper does not concern with the effective thermal expansion coefficients, which will be treated later.

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